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Approximate Laplace Transform Inversions in Viscoelastic Stress Analysis

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An investigation was made of approximate methods for inverting Laplace transforms that occur in viscoelastic stress analysis when use is made of the elastic-viscoelastic analogy. Alfrey's and ter Haar's methods and Schapery's direct method were examined and shown to be special cases of a general inversion formula due to Widder. Schapery's least squares method and several techniques based on orthogonal function theory were also examined. Viscoelastic solutions to two problems involving deformations and stresses in solid propellant rocket motors under axial and transverse acceleration loads were obtained by use of several of the methods discussed. The problems were typical of the type where the associated elastic solution is known only numerically. The use of the orthogonal polynomial methods is explained in detail, and their limitations discussed. From the investigation described, it was concluded that Schapery's direct method and ter Haar's method generally give good results when applicable. Widder's general inversion formula, which includes Alfrey's method as a special case, is not useable for the type problems of interest here. Although the orthogonal polynomial methods possess characteristics that make them especially suited to the type problems considered, their use appears limited by severe computational difficulties. Schapery's least squares method gives good results to most problems of interest.

Introduction

The elastic-viscoelastic analogy has been the method most commonly used for obtaining stress distributions and displacements in linear viscoelastic bodies. The analogy has been derived using both a separation of variables technique and an integral transform method. Alfrey[1] was the first to formulate the analogy using the separation of variables technique, whereas Read[2] was first to derive the analogy using Fourier integral transforms. Lee[3] subsequently showed that the analogy could also be derived by use of the Laplace transform. Further developments and extensions have been concisely summarized by Hilton.[4] In the remarks that follow, the elastic-viscoelastic analogy discussed will be the form derived by Lee.[3]

The elastic-viscoelastic analogy can be shown to exist by operating on the field equations, constitutive equations, and boundary conditions of a linear viscoelastic body with the Laplace transform with respect to time. This operation reduces derivatives and integrals with respect to time to algebraic expressions of the transform parameter. The equations that result after this operation are analogous to the field equations, constitutive equations, and boundary conditions that govern the behavior of an elastic body of the same geometry as the viscoelastic body. If the solution to this associated elastic problem can be obtained, the solution to the time dependent viscoelastic problem can be obtained by operating upon the stresses and displacements of the associated elastic problem with an inverse Laplace transform. Thus, the results of the theory of elasticity can be used to obtain solutions to viscoelastic problems.

There are three general classes of problems where an exact form of the elastic-viscoelastic analogy does not apply. The first class consists of problems with moving boundaries where the motion is due to some external source and is entirely different from the usual infinitesimal deformations due to load, temperature, etc. The second class consists of problems where the type of boundary condition at a boundary point changes with time, i.e., a stress boundary condition at a point at one instant of time changes to a displacement boundary condition at another instant of time and vice versa. The third class consists of problems where the compressible material properties are time dependent. If the material properties are time dependent, the differential equations have variable coefficients or the integral equations are not of the convolution type. Thus, the constitutive equations do not reduce to algebraic expressions when operated on with the Laplace (or Fourier) transform.

Although these limitations are serious and exclude from consideration many problems of interest in viscoelastic stress analysis of solid propellant rocket motors, there are still many problems that fall within the realm of application of the analogy which are of interest. Until recently, viscoelastic solutions to problems in which the associated elastic problem was obtained by some numerical technique have been unobtainable by this method because of the difficulty of taking the inverse Laplace transform of functions that are known

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only numerically, i.e., functions known only at discrete points in the body and only for discrete values of the material constants. Also, there exist many problems that possess associated elastic solutions that are algebraically too complicated to be readily inverted by formal inversion methods.

In this paper, several methods for accomplishing the approximate Laplace transform inversion of a function known only numerically for discrete values of the transform parameter will be discussed. Since an algebraically complicated associated elastic solution can be reduced to a solution known only at discrete points and for discrete values of the material constants, only the numerically known solution will be discussed.

Numerical Transformed Viscoelastic Solution

At this point it appears beneficial to outline the procedure for obtaining transformed viscoelastic solutions from associated elastic solutions obtained by some numerical process.

When the transformed viscoelastic solution or associated elastic solution can be expressed analytically, the moduli of the associated elastic problem are functions of the transform parameter. The form of these equivalent moduli depends on the transformed viscoelastic moduli. The boundary conditions of the associated problem also depend on the transform parameter. Laplace transform inversion of the associated elastic problem yields the viscoelastic solution directly.

When the associated elastic solution is not available in analytical form suitable for formal inversion, numerical values of the transformed viscoelastic solution must be obtained which correspond to certain discrete values of the transform parameter. If the time variation of the viscoelastic moduli of the particular material of interest is known, it is possible to express the equivalent moduli of the material in terms of the Laplace transform of the viscoelastic moduli. Since the equivalent moduli and the transformed boundary conditions can be evaluated for particular values of the transform parameter, the associated elastic solution can be evaluated for particular values of the transform parameter. The transformed viscoelastic solution (associated elastic solution) then can be found for all real values of the transform parameter. Now, if a method of inverting the solution for discrete values of the transform parameter is available, the viscoelastic solution to the problem can be obtained.

To further illustrate this procedure, consider the problem of finding the axial displacements in a finite-length, hollow, circular, linear viscoelastic cylinder bonded to a rigid case on the outer boundary of the cylinder and on one end and subjected to an axial acceleration loading. Application of the elastic-viscoelastic analogy involves the Laplace transform inversion of the solution of an associated elastic problem which consists of an elastic cylinder of the same shape subjected to boundary and loading conditions that are equal

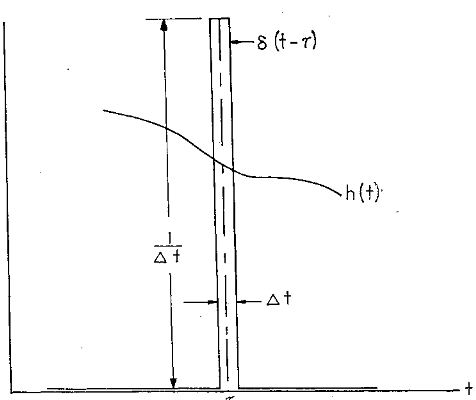

Fig. 1 Illustration of sifting property of Dirac delta function.

to the transformed viscoelastic conditions. Inversion of such a solution would then yield the solution to the viscoelastic problem.

No closed form solution for this associated elastic problem has yet appeared in the literature, although a numerical solution has been obtained by Parr.[5] This solution utilized an iterative process to solve a system of finite-difference equations to determine stress functions at a network of discrete points in the body. The stresses, strains, and displacements were evaluated from the stress functions by finite-difference forms of various analytical relations.

The axial displacement w at any point in the elastic problem could be expressed as

$$w = (b\delta g/E)f[(l/2b), (a/b), \nu] \tag{1}$$

where

b = outer radius of propellant
a = inner radius of propellant
l = length of propellant grain
δ = density of propellant
g = axial acceleration
E = tensile modulus
ν = Poisson's ratio

Only three input parameters were required for the solution, the radius ratio a/b, the length-to-diameter ratio $l/2b$, and Poisson's ratio ν. The transformed viscoelastic solution is then obtained from Eq. (1) by replacing the elastic constants E and ν by the equivalent viscoelastic constants $E^\#(p)$ and $\nu^\#(p)$ and replacing the acceleration g by the transform of the acceleration in the viscoelastic problem $\bar{g}(p)$. The transformed viscoelastic displacement at a point is then given by

$$\bar{w}(p) = \frac{b\delta\bar{g}(p)}{E^\#(p)} f\left(\frac{l}{2b}, \frac{a}{b}, \nu^\#(p)\right) \tag{2}$$

$E^\#(p)$ and $\nu^\#(p)$, the equivalent moduli, are defined as

$$E^\#(p) = p\bar{E}^*(p)$$
$$\nu^\#(p) = p\bar{\nu}^*(p) \tag{3}$$

where $\bar{E}^*(p)$ and $\bar{\nu}^*(p)$ are the Laplace transforms of the time-dependent relaxation modulus in uniaxial extension $E^*(t)$ and the viscoelastic Poisson's ratio $\nu^*(t)$ determined from a uniaxial tensile test, respectively.

The procedure for obtaining the transformed viscoelastic displacement $\bar{w}(p)$ at various values of p is as follows. First, compute values of $\bar{g}(p)$, $E^\#(p)$, and $\nu^\#(p)$ which correspond to a particular value of p, e.g., $p = p_0$. Second, the numerical procedure for obtaining the elastic solution w, as stated in Eq. (1), is then used with $\nu = \nu^\#(p_0)$, $E = E^\#(p_0)$, and $g = \bar{g}(p_0)$ (also values of a, b, l, and δ). This operation is equivalent to the procedure for evaluating Eq. (2). $\bar{w}(p_0)$ is therefore obtained by evaluating the elastic solution for w. Repeating this procedure with $p = p_i$, $i = 1, 2, 3, \ldots,$ gives values of the transformed function for various values of p_i. Thus, to obtain the viscoelastic solution, the values of the $\bar{w}(p_i)$, $i = 1, 2, 3, \ldots,$ must be used to obtain the inverse function $w(t)$. Several techniques for accomplishing such an inversion approximately will be discussed in the following section.

Approximate Laplace Transform Inversion Methods

General Theory

Before beginning a discussion of the approximate inversion methods, certain elementary aspects of Laplace transform theory will be reviewed here for terminology and completeness.†

† For a comprehensive treatment of Laplace transform theory, see one of the standard treatise on the subject such as Widder.[6]

The Laplace transform $\bar{f}(p)$ of a function $f(t)$ is commonly indicated by $L\{f(t)\}$ and defined as

$$L\{f(t)\} = \bar{f}(p) = \int_0^\infty f(t)e^{-pt}dt \tag{4}$$

where p is the transform parameter and $\bar{f}(p)$ is a function of p different from, but related to $f(t)$. In order for the Laplace transform of a function to exist, it is sufficient that the function be sectionally continuous in every finite interval in the range $t \geq 0$ and that the function be of exponential order as $t \to \infty$. The parameter p, in general, is a complex variable.

If the transformed function $\bar{f}(p)$ is analytic and of order p^{-c} in a half-plane $Re(p) \geq \alpha$, where c is a constant greater than zero, then the inverse Laplace transform gives the original function $f(t)$, termed the "indicial function," in terms of a complex integral as

$$L^{-1}\{\bar{f}(p)\} = f(t) = \frac{1}{2\pi i} \lim_{\beta\to\infty} \int_{\alpha-i\beta}^{\alpha+i\beta} \bar{f}(p)e^{pt}dp \tag{5}$$

Here $L^{-1}\{f(p)\}$ is used to indicate the inverse Laplace transform of $f(t)$. Although this inversion integral, which is an integral in the complex plane along a line parallel to the imaginary axis, can be written in terms of a real infinite integral, the evaluation is more difficult than evaluation of the complex integral. Hence, the complex form is usually used in conjunction with certain auxiliary line integrals and the theory of residues. In any case, the inversion can be quite difficult and tedious for complicated forms of the transformed function $\bar{f}(p)$. If the function $\bar{f}(p)$ is known only numerically for discrete values of p, some approximate inversion scheme must be used.

The need for inverting Laplace transforms has been experienced in many fields, and, as a result, approximate inversion methods have been developed in connection with several distinct subjects. A unified treatment of the most promising approximate inversion methods is presented in the following sections.

Inversion Methods Based on Widder's General Inversion Formula

Widder[6] has suggested use of a general inversion formula first proposed by Post.[7] The general inversion formula represents the general case of a set of inversion methods that have recently been considered, namely, the methods of Alfrey,[1] ter Haar,[8] and Schapery.[9]

Widder's general inversion formula

Widder's general inversion formula is based on the sifting property of the Dirac delta function. This property is best stated by the formula

$$\int_0^\infty h(t)\delta(t-\tau)dt = h(\tau) \qquad \tau > 0 \tag{6}$$

where

$$\delta(t-\tau) = \begin{cases} 0 & t \neq \tau \\ \lim\limits_{\Delta t\to 0} \quad 1/\Delta t & t = \tau \end{cases}$$

A graphical interpretation of the sifting property of the delta function is shown in Fig. 1. From the figure it can be seen that $h(t)$ is essentially constant in the region where the impulse function is large ($t = \tau$), and the product $h(t)\cdot\delta(t-\tau)$ is zero elsewhere. The integral in Eq. (6) is accordingly equal to $h(t)|_{t=\tau}$ times the area below the $\delta(t-\tau)$ curve. Since, by definition, the area is unity, i.e.,

$$\int_0^\infty \delta(t-\tau)dt \equiv 1 \qquad \tau > 0 \tag{7}$$

the result expressed in Eq. (6) becomes apparent.

To obtain the Laplace transform in a form in which this relation may be used, the definition integral is differentiated under the sign of integration n times with respect to the transform parameter p. This operation gives

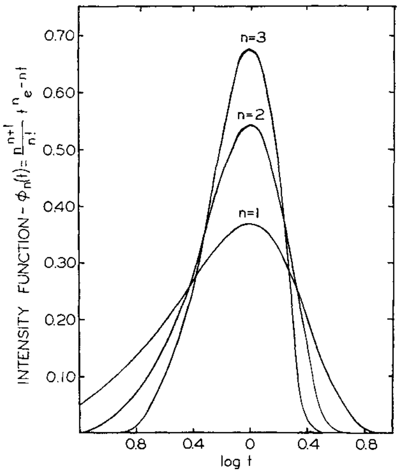

Fig. 2 Intensity functions arising in Widder's general inversion formula.

$$\frac{d^n\bar{f}(p)}{dp^n} = (-1)^n \int_0^\infty f(t)[t^n e^{-pt}]dt \tag{8}$$

With a slight modification, this integral can be converted into a sifting integral as in Eq. (6). The function in brackets resembles a delta function and can be approximated by such a function after the normalization requirements stated in Eq. (7) are satisfied. Since the integral

$$\int_0^\infty t^n e^{-pt}dt = \frac{n!}{p^{n+1}} \tag{9}$$

is not unity, the function in brackets, hereafter called the intensity function, must be multiplied by $p^{n+1}/n!$ before being replaced by the delta function. Then,

$$(p^{n+1}/n!)t^n e^{-pt} \approx \delta(t-t_0) \tag{10}$$

where t_0 is any specified time at which the δ function is not zero. The function on the left has a maximum at $t = n/p$, and if the delta function is assumed to be located at this point, Eq. (10) becomes

$$\delta[t-(n/p)] \approx (p^{n+1}/n!)t^n e^{-pt} \tag{11}$$

The family of functions represented by successively higher values of n is shown in Fig. 2 where the location of the maximum point of the functions has been chosen to be at $t_0 = 1$ or $n/p = 1$. It can be seen from Fig. 2 that increasing the value of n increases the accuracy of the approximation stated in Eq. (11), and with the use of Stirling's formula it can be shown that

$$\lim_{n\to\infty} (p^{n+1}/n!)t^n e^{-pt} = \delta[t-(n/p)]$$

If Eq. (11) is substituted into Eq. (8), there results the sifting integral

$$(-1)^n \frac{p^{n+1}}{n!} \frac{d^n}{dp^n} \bar{f}(p) \approx \int_0^\infty f(t)\delta\left(t-\frac{n}{p}\right)dt \tag{12}$$

Applying the properties of the sifting integral [see Eq. (6)] to Eq. (12) yields the relation

$$(-1)^n \frac{p^{n+1}}{n!} \frac{d^n}{dp^n} \bar{f}(p) \approx f(t)|_{t=n/p} \tag{13}$$

Equation (13) can be written in a slightly different form to give the general inversion formula as proposed by Widder. This relation is

$$f(t) = \lim_{n\to\infty} \left[\frac{(-1)^n p^{n+1}}{n!} \frac{d^n}{dp^n} \bar{f}(p)\right]_{p=n/t} \tag{14}$$

where the limit has been applied to indicate that the accuracy of the approximation increases with n because of the stronger resemblance of the intensity function to the delta function for higher values of n.

Alfrey's approximation

Alfrey's approximation is

$$f(t) \approx \left[-p^2 \frac{d\bar{f}(p)}{dp}\right]_{p=1/t} \tag{15}$$

By inspection of Eq. (14), it can be seen that Alfrey's approximation is exactly the first-order approximation of Widder's general inversion formula, i.e., Widder's formula when $n = 1$. Hence, the assumptions and limitations of Alfrey's inversion method are a special case of those of Widder's method.

ter Haar's approximation

ter Haar[8] proposed a method for approximate inversion of the Laplace transform which did not involve knowing any derivatives of the transformed function. ter Haar's approximation is

$$f(t) \approx [p\bar{f}(p)]_{p=1/t} \tag{16}$$

To arrive at this relation, a slightly different approach was used by ter Haar. Instead of differentiating the definition integral, as was done previously, ter Haar simply multiplied and divided the integrand of this integral by t. Performing this operation on Eq. (4) results in the expression

$$\bar{f}(p) = \int_0^\infty \frac{f(t)}{t}\,[te^{-pt}]dt \tag{17}$$

As was pointed out by ter Haar, the function in brackets resembles a delta function and therefore may be approximated by the function

$$te^{-pt} \approx [\delta(t - t_0)/p^2] \tag{18}$$

where the factor p^2 has been introduced for normalization purposes. Making this substitution in Eq. (17) yields

$$p^2\bar{f}(p) \approx \int_0^\infty \frac{f(t)}{t}\,\delta(t - t_0)dt \tag{19}$$

It can be seen by comparing Eqs. (18) and (10) that the intensity function of ter Haar, which is assumed to be approximated by a delta function, is the same as that of Widder when $n = 1$. Thus, the nature of ter Haar's approximation is the same as that of Widder when $n = 1$, although the indicial functions in Eqs. (19) and (12) differ by a factor of $1/t$. This difference results in somewhat different approximations.

Applying the sifting integral properties to Eq. (19) gives

$$p^2\bar{f}(p) \approx [f(t)/t]_{t=t_0} \tag{20}$$

Assuming the delta function location t_0 to be at the point where the intensity function is a maximum ($t = 1/p$) allows Eq. (20) to be rewritten in the form of Eq. (16), which is ter Haar's approximation.

Schapery's direct method of approximation

Schapery's direct method[9] of approximate inversion gives the same result as ter Haar's except that the location of the delta function is chosen differently which results in a different relation between p and t. To obtain Schapery's formula, the delta function is located at the centroid of the intensity function in the log$t$ scale which is the point where $pt \approx 0.5$. The final form of Schapery's direct method of approximation is

$$f(t) = [p\bar{f}(p)]_{p=0.5/t} \tag{21}$$

Schapery points out that the method is mainly limited to problems where $p\bar{f}(p)$ is essentially a linear function of log$p$.

Method Based on Principle of Least Squares

This method was first proposed by Schapery.[9] Schapery concludes from his studies of irreversible thermodynamics that the class of problems to which the elastic-viscoelastic analogy may be applied has time-dependent solutions of the form

$$f(t) = C_1 + C_2t + \theta(t) \tag{22}$$

where C_1 and C_2 are constants and $\theta(t)$ is the transient component of the solution defined as

$$\theta(t) = \int_0^\infty H(\tau)e^{-t/\tau}d\tau \tag{23}$$

The function $H(\tau)$, referred to as a spectral function, may consist either entirely or partly of Dirac delta functions. As Schapery[9] points out, if $H(\tau)$ is the sum of a series of delta functions such as

$$H(\tau) = \sum_{i=1}^{m} h_i(\tau - \tau_i) \tag{24}$$

then the transient component of the response function is

$$\theta(t) = \sum_{i=1}^{m} h_i e^{-t/\tau_i} \tag{25}$$

The form of Eq. (25) suggests that $\theta(t)$ may be expressed approximately as

$$\theta(t) \approx \theta^*(t) = \sum_{i=1}^{m} g_i e^{-t/\alpha_i} \tag{26}$$

where the α_i and g_i are constants to be determined by minimizing the mean square error between $\theta(t)$ and $\theta^*(t)$. This mean square error is, by definition,

$$E^2 = \int_0^\infty [\theta(t) - \theta^*(t)]^2dt \tag{27}$$

By prescribing the α_i, the g_i may be determined by minimizing E^2 with respect to g_i. This minimization process results in the expression

$$\frac{\partial E^2}{\partial g_i} = \int_0^\infty 2[\theta(t) - \theta^*(t)]e^{-t/\alpha_i}dt = 0 \qquad (i = 1, 2, \ldots, m) \tag{28}$$

or, equivalently,

$$\int_0^\infty \theta(t)e^{-t/\alpha_i}dt = \int_0^\infty \theta^*(t)e^{-t/\alpha_i}dt \qquad (i = 1, 2, \ldots, m) \tag{29}$$

Equation (29) essentially means that, for the mean square error of the approximation to be a minimum, the Laplace transform of the approximation must equal the Laplace transform of the exact function at least at the m points $p = 1/\gamma_i$, $i = 1, 2, \ldots, m$. Therefore, m relations are available relating the Laplace transforms of $\theta(t)$ and $\theta^*(t)$, each evaluated at $p = 1/\alpha_i$, i.e.,

$$\bar{\theta}(p)|_{p=1/\alpha_i} = \bar{\theta}^*(p)|_{p=1/\alpha_i} \qquad (i = 1, 2, \ldots, m) \tag{30}$$

Since $\theta(t)$ was assumed to be of the form indicated in Eq. (26), Eq. (30) reduces to the series of equations:

$$\bar{\theta}(p)|_{p=1/\alpha_i} = \sum_{j=1}^{m} \frac{g_j}{[(1/\alpha_i) + (1/\alpha_j)]} \qquad (i = 1, 2, \ldots, m) \tag{31}$$

which may be used to evaluate the g_i in terms of the values of the transformed function $\bar{\theta}(p)$ at the points $p_i = 1/\alpha_i$. Thus, the transient component of the solution of $\theta(t)$ may be determined and the constants C_1 and C_2 determined from given initial and boundary conditions.

Papoulis' Legendre Polynomial Inversion Method

Expansion of the desired time-dependent indicial function $f(t)$ into a series of orthogonal polynomials in real time as a means of approximate inversion has received the attention of two investigators, Papoulis[10] and Lanczos.[11] The methods are based on the idea that the form of the transformed polynomials is known and the coefficients of the polynomials are calculable in terms of values of the transformed function at equidistant points along the real axis. The method, as described by Papoulis,[10] which uses Legendre polynomials for the orthogonal set is presented in the following section.

As described by Papoulis,[10] the definition integral [Eq. (4)] may be transformed according to the relation

$$e^{-\gamma t} = x \qquad \gamma > 0 \tag{32}$$

where γ is a scaling factor that remains constant throughout any particular expansion; γ will be discussed in more detail later after it becomes clear what purpose it serves. The function $f(t)$ in Eq. (4) becomes

$$f(t) = f[-(1/\gamma)\ln x] \equiv \hat{f}(x) \tag{33}$$

Making these substitutions, Eq. (4) becomes

$$\gamma\bar{f}(p) = \int_0^1 x^{(p/\gamma)-1}\hat{f}(x)dx \tag{34}$$

By expressing p in terms of γ as

$$p = (2k+1)\gamma \tag{35}$$

Eq. (34) takes the form

$$\gamma\bar{f}(2k\gamma + \gamma) = \int_0^1 x^{2k}\hat{f}(x)dx \tag{36}$$

Therefore, if the value of $\bar{f}(p)$ is known at the point $p = (2k\gamma + \gamma)$, this is equivalent to knowing the $2k$th moment of the function $\hat{f}(x)$ in the interval (0, 1).

The Legendre polynomials $P_n(x)$ form a complete orthogonal set in the interval $(-1, 1)$ and by extending the definition of $\hat{f}(x)$ in the interval $(-1, 1)$ by defining $\hat{f}(x)$ to be an even function, i.e.,

$$\hat{f}(x) = \hat{f}(-x) \tag{37}$$

the function $\hat{f}(x)$ can be expanded into a series of even Legendre polynomials. Such an expansion gives

$$\hat{f}(x) = \sum_{j=0}^{\infty} C_j P_{2j}(x) \tag{38}$$

The major task remaining is to evaluate the C_j. It is known[10] that the series expansion of a function is uniquely determined by its moments that are known quantities as seen from Eq. (36). Using the fact that $P_{2j}(e^{-\gamma t})$ is an even polynomial in $e^{-\gamma t}$ of degree $2j$, Papoulis shows that the transform of $P_{2j}(e^{-\gamma t})$ is of the form

$$\overline{P_{2j}}(p) = \frac{[p-\gamma][p-3\gamma]\ldots[p-(2j-1)\gamma]}{p[p+2\gamma]\ldots[p+2j\gamma]} \tag{39}$$

Therefore, taking the Laplace transform of both sides of Eq. (38) gives

$$\bar{f}(p) = \frac{C_0}{p} + \sum_{j=1}^{\infty} \frac{[p-\gamma]\ldots[p-(2j-1)\gamma]}{p\ldots[p+2j\gamma]}C_j \tag{40}$$

Replacing p first by γ then 3γ, 5γ, . . . , $(2j+1)\gamma$, . . . in Eq. (40) gives the following system of equations to evaluate the C_j:

$$\gamma\bar{f}(\gamma) = C_0 \qquad \gamma\bar{f}(3\gamma) = \tfrac{1}{3}C_0 + \tfrac{2}{15}C_1$$

$$\gamma\bar{f}(2j\gamma + \gamma) = \frac{C_0}{2j+1} + \frac{2jC_1}{[2j+1][2j+3]} + \ldots + \frac{2j[2j-2]\ldots 2C_j}{[2j+1][2j+3]\ldots[4j+1]} \tag{41}$$

Thus, using Eqs. (41) to evaluate the C_j in terms of the known values of $\bar{f}(p)$ determines the expansion defined in Eq. (38), which in terms of real time is

$$f(t) = \sum_{j=0}^{\infty} C_j P_{2j}(e^{-\gamma t}) \tag{42}$$

As seen from Eq. (42), the scaling factor γ enters the final expression as a factor of the exponent of e. The function e^{-x} varies over the interval $(-2 < \log x < 1)$ but is essentially constant elsewhere. If a function $f(t)$ varies over an interval greater than $(-2 < \log\gamma t < 1)$, then functions of the form $e^{-\gamma t}$ cannot adequately describe its variation over the total range of variation. Thus, γ must assume multiple values depending on the interval of time of interest and different expansions must be obtained for each choice of γ.

Thus, the inversion of the Laplace transform can be accomplished by expanding the desired indicial function $f(t)$ in terms of an infinite convergent series of orthogonal functions whose coefficients are calculable in terms of equidistant values of the Laplace transform along the real axis.

Lanczos method,[11] which utilizes Legendre polynomials, differs in theory only slightly from the method described by Papoulis. Instead of using the exact form of the polynomials used by Papoulis, Lanczos uses a slightly modified form for the polynomials, which are orthogonal on the interval (0, 1). Lanczos has refined the method of application to the point that computational efforts are minimized, and the coefficients C_j are determined in a slightly different form than in Papoulis' method. The nature of the results of the two methods are very similar.

The basic idea of the orthogonal polynomial approximations has also been extended by Papoulis and Lanczos to include Chebycheff and Laguerre polynomials. For a comprehensive description of these methods, see Ref. 12.

Numerical Examples

The methods of approximate inversion of the Laplace transform just presented vary considerably in regard to the assumptions and mathematical techniques used in their derivation. Also, application of the methods is considerably different. In order to illustrate the use of and clarify the restrictions on the various methods, solutions to two problems are presented which are of interest in solid propellant structural integrity analysis. The problems are typical of the type where the solution is unobtainable by formal Laplace transform inversion methods. The solutions presented here are intended only to indicate the method of application and nature of the results of the various methods and not necessarily to indicate the accuracy obtainable with any method.

The first problem is that of finding the circumferential stress σ_θ at a point on the inner boundary of an infinite-length, hollow, circular cylinder of linear viscoelastic material. The cylinder is bonded to an elastic case and is lying in a horizontal position loaded by its own weight. The cylinder is assumed supported by a concentrated load as illustrated in Fig. 3.

The second problem involves the determination of the axial displacement at a point on the inner boundary of a finite-length, hollow, circular cylinder of linear viscoelastic material bonded to a rigid case and subjected to an axial acceleration load as shown in Fig. 4.

In solving both problems, realistic material property data were used. The actual material behavior is idealized but is typical of many solid propellant materials. It is characterized by the tensile relaxation modulus $E^*(t)$ shown in Fig. 5 and is seen to have relaxation processes occurring over several decades of logarithmic time. The collocation method[13] for fitting a series of exponentials to measured data was used to obtain an analytical expression for the tensile relaxation

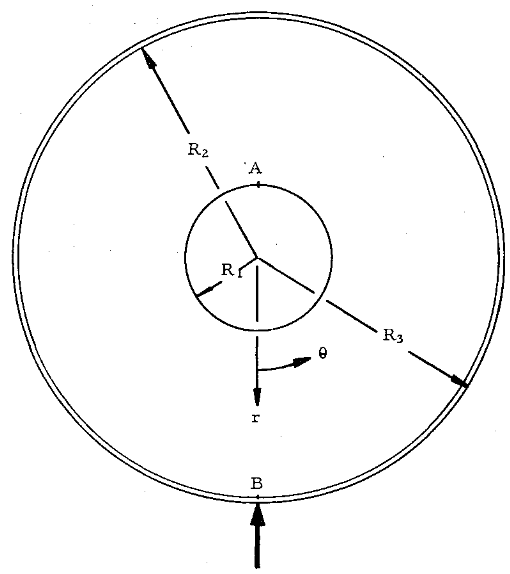

Fig. 3 Infinite-length cylinder under transverse acceleration load supported by point load at *B*.

modulus. The resulting expression for $E^*(t)$ used to obtain the curve in Fig. 5 is

$$E^*(t) = 600 - 1945.298e^{-500t} + 16334.19e^{-50t} + 21670.066e^{-5t} + 16951.871e^{-0.5t} + 6425.041e^{-0.05t} \quad (43)$$

where the units of $E^*(t)$ and t are pounds per square inch and minutes, respectively.

The material was assumed to be elastic in isotropic tension and the time dependent Poisson's ratio[14] is thus expressible as

$$\nu^*(t) = \tfrac{1}{2} - [E^*(t)/6K] \quad (44)$$

where the bulk modulus K was assumed to have the value of 500,000 psi.

Problem I: Transverse Acceleration of Infinite-Length Cylinder

A viscoelastic solution has been obtained by Lianis[15] to the problem of transverse slump of a solid propellant rocket grain bonded to a rigid case, but, as yet, no solution has appeared in the literature to the problem of a solid propellant rocket grain, assumed to be viscoelastic, bonded to an elastic case. From the results of the elastic solution of Gillis,[16] the affect of the flexible case is considerable and would also be significant for a viscoelastic analysis.

The elastic solution obtained by Gillis[16] was used to obtain the viscoelastic solution to the problem of transverse acceleration. Gillis solved the problem of an infinite-length, hollow, circular, elastic cylinder bonded to an elastic case. The

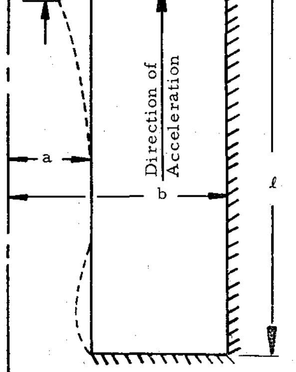

Fig. 4 Finite-length cylinder under axial acceleration load.

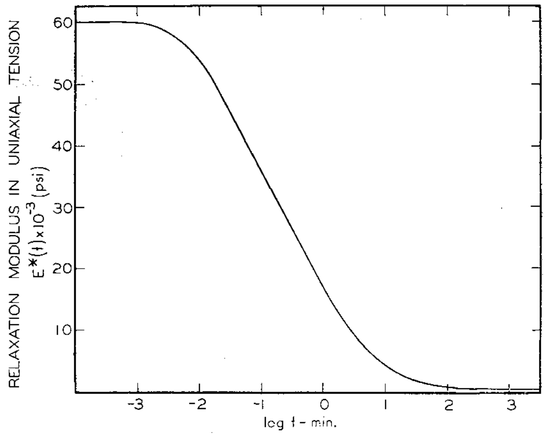

Fig. 5 Viscoelastic moduli used in numerical examples.

method of solution involved application of Muskhelishvili's complex variable method for solving problems with circular boundaries. Each concentric cylinder, i.e., the case and the grain, was considered separately. The unknown interface stresses were represented as infinite complex Fourier series with unknown coefficients. These coefficients were later evaluated by equating the normal and shearing stresses and the displacements at the interface.

The solution resulting from this technique was in the form of an infinite series whose coefficients were algebraically complicated functions of the elastic material properties; hence, formal Laplace transform inversion of such a form was impractical. Therefore, the approximate Laplace transform inversion methods described herein are particularly well suited for this problem.

The geometry of the problem considered is as shown in Fig. 3, and the following numerical values were used for the various descriptive parameters; $R_1/R_2 = 0.25$, $R_3/R_2 = 1.005$, $\rho/\delta = 4.733$, $\nu_c = 0.3$, and $G_c = 30.0 \times 10^6$ psi, where ρ is the weight density of case, δ the weight density of propellant, ν_c the Poisson's ratio of elastic case, and G_c the modulus of elasticity of case in shear.

The dimensionless circumferential elastic stress σ_θ at point A (Fig. 3) is a function of Poisson's ratio and the elastic tensile modulus. Application of the elastic-viscoelastic analogy to this function in the manner described earlier yields the following expression for the transformed viscoelastic stress:

$$\frac{\bar{\sigma}_\theta(p)}{\delta_0 R_2} = \frac{1}{p} f(\nu^\#, E^\#) \quad (45)$$

The gravitational acceleration has been assumed to be a constant applied at time $t = 0$ so that $\bar{\delta}(p) = \delta_0/p$ and $\bar{\rho}(p) = \rho_0/p$.

The equivalent modulus $G^\#(p)$ is expressible in terms of the equivalent moduli $\nu^\#(p)$ and $E^\#(p)$ as

$$G^\#(p) = \frac{E^\#(p)}{2[1 + \nu^\#(p)]} \quad (46)$$

Repeated computation of $[\bar{\sigma}_\theta(p)/\delta_0 R_2]$ with $\nu = \nu^\#(p)$ and $E = E^\#(p)$ then allows the transformed quantity to be determined numerically for various values of p.

The transformed viscoelastic solution will now be inverted by two of the methods discussed earlier.

Schapery's direct method

Application of Schapery's direct method of inversion is straightforward and needs no elaboration regarding its use. Equation (21) can be readily applied to this problem to give the direct method approximation as

$$\frac{\sigma_\theta(t)}{\delta_0 R_2} = \left[\frac{p\bar{\sigma}_\theta(p)}{\delta_0 R_2}\right]_{p=0.5/t} \quad (47)$$

It is seen that the time-dependent function $\sigma_\theta(t)$ is thus numerically equal to the transformed function $p\bar{\sigma}_\theta(p)$ when $p = 0.5/t$. Values of $p\bar{\sigma}_\theta(p)$ were computed for various p values in the manner described earlier maintaining as much accuracy as possible on an IBM 7090 computer. The results from these computations are shown in Fig. 6 on the appropriate curve.

Papoulis' Legendre polynomial approximation

Of the methods discussed in this paper, the methods based upon properties of orthogonal polynomials appear to be the least known. Therefore, more detail will be included in application of these methods than would normally be needed. The inversions performed here are intended only to indicate the method of application and the nature of the results obtained and not necessarily to indicate the best accuracy obtainable with these methods.

The first decision to be made is how many terms should be included in the expansion and what values the scaling factor γ should assume. In order to cover a span of time of several decades, a very large number of terms would have to be included in the series expansion for the time-dependent function, Eq. (42). An alternative to including a large number of terms in the series is to develop several expansions all with a smaller number of terms but which apply in different intervals of time. This is accomplished by considering expansions with different values of the scaling factor γ used in each expansion.

Previous results have shown[12, 17] that, although Schapery's direct method may be in error a significant amount at any instant of time, the method, if applicable, still indicates the general behavior of the time-dependent function. Thus, Schapery's direct method can be used as a guide in determining what intervals of time are of greatest interest when deciding what values of the scaling factor γ are to be used. For the problem under consideration, observation of the results of the direct method (Fig. 6) indicates that the time interval of greatest interest is between $\log t = -3$ and $\log t = +3$, since the time-dependent function is essentially constant elsewhere.

The choice of the scaling factor γ also depends upon the number of terms considered in the series expansions. For minimum computation effort, it was decided to include only the first six terms in the series expansion. The resulting expression for the time-dependent function then becomes an expression of the form

$$f(t) = \sum_{i=0}^{5} A_i e^{-2i\gamma t} \tag{48}$$

where the A_i are constants to be determined. Observation of the time dependence of these exponentials indicates that the series will be applicable, at most, over an interval of about two decades of logarithmic time. These limits on the interval of applicability are

$$-2.0 \leq \log\gamma t \leq 0$$

or

$$-[2.0 + \log\gamma] \leq \log t \leq -[\log\gamma] \tag{49}$$

Thus, by varying γ it is seen that the intervals of time over which the expansions are applicable are changed, and γ should be selected accordingly. From considerations of this type, it was determined that, for the problem under consideration, expansions would be made with $\gamma = 2.0$, $\gamma = 0.1$, and $\gamma = 0.01$.

Normal application of this Legendre polynomial inversion method involves solving Eqs. (41) for the C_j which, in this case, are evaluated in terms of $[p\bar{\sigma}_\theta(p)/\delta_0 R_2]$, where p assumes the successive values γ, 3γ, 5γ, 7γ, 9γ, and 11γ. Substitution of these C_j into Eq. (42) then gives the desired time-

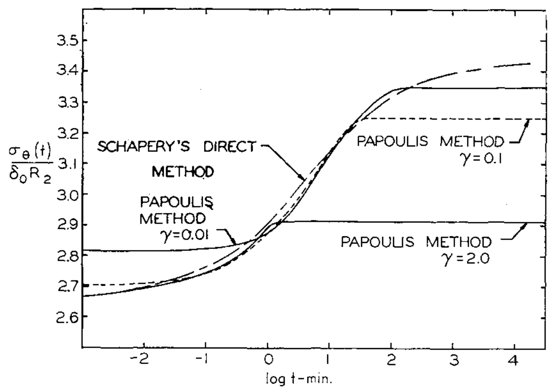

Fig. 6 Circumferential stress in infinite-length cylinder subjected to transverse acceleration load.

dependent expression. For convenience in applying the inversion method, it can be easily programed for performance on a high-speed computer. This was done, and the resulting time-dependent solution was obtained as a function of the values of the transformed function. Expressions for the time-dependent function, where up to six terms have been included in the expansion, are included in the Appendix for reference purposes. Here, it is only necessary to say that application of the method yielded the expression

$$[\sigma_\theta(t)/\delta_0 R_2] = A_0 + A_1 e^{-2\gamma t} + A_2 e^{-4\gamma t} + A_3 e^{-6\gamma t} + A_4 e^{-8\gamma t} + A_5 e^{-10\gamma t} \tag{50}$$

where the A_i are given for the various γ values in Table 1.

These equations are shown graphically in Fig. 6 along with the results of Schapery's direct method.

Problem II: Axial Slump of Finite-Length Cylinder

The geometry of the problem under consideration here is shown in Fig. 4, and the descriptive parameters for this problem take the numerical values $a/b = 0.2$ and $l/2b = 2.0$. The problem was discussed earlier in this paper in detail in regard to the method of obtaining the transformed viscoelastic solution from associated elastic solutions obtained by some numerical technique. Equation (2) gives the expression for the transformed viscoelastic displacement in terms of the equivalent moduli and boundary conditions. Assuming the acceleration $g(t)$ to be a step function of magnitude g_0 applied at time $t = 0$ allows Eq. (2) to become

$$\frac{\bar{w}(p)}{b\delta g_0} = \frac{f[(l/2b), (a/b), \nu^{\#}(p)]}{E^{\#}(p)} \tag{51}$$

This expression was used to compute $[\bar{w}(p)/b\delta g_0]$ at various p values and will now be inverted by ter Haar's approximate inversion formula and Papoulis' Legendre polynomial inversion method.

In problem I, all of the values for the transformed functions were computed by use of a digital computer. Since this involves calculating the associated elastic solution for each p value of interest, it can sometimes require large amounts of computer time. In order to determine if the

Table 1 Coefficients for Legendre polynomial series expansion for circumferential stress in infinite-length cylinder

	$\gamma = 2.0$	$\gamma = 0.1$	$\gamma = 0.01$
A_0	+2.9101288	+3.2446153	+3.3485413
A_1	−1.1336263	−1.5955523	−0.1508337
A_2	+4.9700432	+7.0635583	+1.3693328
A_3	−11.220680	−17.294206	−6.7259964
A_4	+11.587841	+19.018909	+10.444514
A_5	−4.514669	−7.7341720	−5.4694445

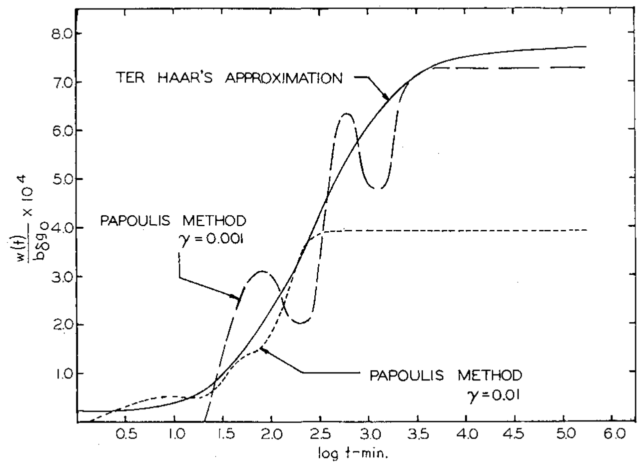

Fig. 7 Maximum vertical deflection of finite-length cylinder under axial acceleration.

associated elastic solution could be calculated for only a few values of p and the results interpolated to obtain the results for any p value of interest, the data for the values of the transformed displacement were obtained from a graph plotted from only a few exact values of the transformed displacement.

ter Haar's approximation

ter Haar's approximation was used to indicate that the method is the same as Schapery's direct method except that the results are shifted an amount equal to log2 parallel to the log$t$ scale. Application of Eq. (16) gives the time-dependent vertical displacement at the location shown in Fig. 4 as

$$\frac{w(t)}{b\delta g_0} = \left[\frac{p\bar{w}(p)}{b\delta g_0}\right]_{p=1/t} \tag{52}$$

The results from Eq. (52) are shown graphically in Fig. 7.

Papoulis' Legendre polynomial inversion method

Expansions for $[w(t)/b\delta g_0]$ were obtained for scaling factor values of $\gamma = 0.001$ and $\gamma = 0.01$ again mainly to indicate the nature of the results. The computer program mentioned previously was used, and the results are expressions of the type

$$[w(t)/b\delta g_0] = A_0 + A_1e^{-2\gamma t} + A_2e^{-4\gamma t} + A_3e^{-6\gamma t} + A_4e^{-8\gamma t} + A_5e^{-10\gamma t} \tag{53}$$

where the A_i assumes the values shown in Table 2. The results from these calculations are shown in Fig. 7.

Discussion

Widder's General Inversion Formula

Widder's method, which includes Alfrey's method as a special case, can be applied to problems having discrete elastic solutions but with doubtful accuracy. Application of these methods involves obtaining derivatives of successively higher orders. Because of the nature of the problem being considered, these derivatives must be obtained by numerical differentiation techniques. In order to examine the practicality of obtaining accurately the derivatives of a function by using various numerical differentiation techniques, several functions whose derivatives were known exactly were examined.[12] The functions examined were relatively well behaved and of exponential and decaying periodic forms. Derivatives up to the third order were obtained with the use of differentiation formulas that involved using three, five, and seven data points to obtain each derivative. It was found that derivatives of functions obtained near the ends of the interval under consideration were not reliable. Third-order derivatives were obtainable within the interval using central difference formulas with results falling within engineering accuracy. However, if the behavior of the function is not clear, the process of obtaining derivatives numerically does not give results having sufficient accuracy to warrant their use. This is due to the influence of the increment size between data points and the formula to be used in obtaining the derivative, i.e., 3, 5, or 7 point formulas. In general, attempting to obtain these derivatives numerically is a questionable procedure and should be avoided if possible. Thus, the applicability of Alfrey's and Widder's methods to problems of this type appears limited.

Table 2 Coefficients for Legendre polynomial series expansion for axial slump of a finite-length cylinder

	$\gamma = 0.01$	$\gamma = 0.001$
A_0	+0.00039179618	+0.00672576811
A_1	−0.002994930	−0.0063681163
A_2	+0.014314120	+0.051678120
A_3	−0.032746425	−0.15256234
A_4	+0.033910939	+0.18186449
A_5	−0.012904986	−0.75686391

Schapery's Direct Method and ter Haar's Method

The direct method of Schapery and the method of ter Haar, being the same except for the relation between p and t, can be considered as one. This method obviously possesses a simplicity that is highly desirable. The method can be applied directly by simply multiplying the value of the transformed function by p and evaluating the result at $p = a/t$. Application of the method introduces no error except that inherent in the method itself. It appears that of the methods considered in this paper, the methods of ter Haar and Schapery are easiest to apply for all problems.

Unfortunately, the methods are not applicable except when the function $p\bar{f}(p)$ is practically a linear function of log$p$. If the functions vary rapidly with time, the methods can give some extremely poor results.[12] For many problems in solid propellant structural integrity analysis, the behavior of the time-dependent function is such that the method is applicable. This allows a determination of an approximate solution very conveniently. However, at any instant of time the answer may be in error, and there is no way to estimate the error involved. If care is exercised in applying the method only to applicable problems, good results can generally be obtained.

Schapery's Least-Squares Method

Schapery's least-squares method has been examined in detail elsewhere[9, 18] and hence was not used to obtain the solution to a specific problem in this paper. The extension suggested by Schapery[9] to obtain better results to the problem of approximately inverting the function $[p\bar{R}(p)/E_G]$ considered by Muki and Sternberg[14] was made with satisfactory results. Eight terms were included in the time-dependent expression. The results are shown in Fig. 8 along with Schapery's direct method approximation, ter Haar's approximation, and the finite-difference solution of Lee and Rogers.[19]

The method is well founded mathematically, and the scheme for collocating the transformed curve gives good results.

Orthogonal Polynomial Inversion Methods

The Papoulis' and Lanczos' methods are essentially the same, and their limitations will be discussed together.

The methods appear to be ideally suited to the inversion of functions known only at discrete real values of the transform parameter p. Fitting the transformed series expansion to the transformed function in the p plane at certain perscribed points allows the coefficients of the expansion to be determined very readily. Unfortunately, the majority of functions of interest in solid propellant structural integrity analysis vary very slowly with time so that the coefficients depend to a large extent on small differences between very large numbers. This leads to computational difficulties and requires the original data, i.e., values of the transformed function, to be known extremely accurately.

Examples of the type of error encountered are shown in Fig. 7 where it will be recalled that the values of the transformed function were obtained by reading values from a graph. From inspection of Fig. 7, it can be seen that the expansions oscillate about some mean value. The magnitude of the oscillation is dependent upon the magnitude of the error involved in the computation.

Illustration of the good approximations that can be made are seen in Fig. 6 where the curves for the various expansions are generally very smooth over their range of application. For $\gamma = 2.0$, the range of application, according to the criterion discussed earlier, i.e., $-2 < \log\gamma t < 0$, is $-2.301 < \log t < -0.301$. It can be seen that the expansion with $\gamma = 2.0$ accurately describes a curve in its range of application. It differs somewhat from the direct method approximation; however, it is not known which of the curves is in error.

Similarly, the expansions for $\gamma = 0.1$ and $\gamma = 0.01$ which apply over the intervals $-1 < \log t < +1$ and $0 < \log t < +2$, respectively, accurately describe the time-dependent function. In regions where the interval of application of two expansions overlap, the expansions seem to agree fairly closely. It can be seen that, by fairing in a curve through the expansion solutions for $\gamma = 2.0$, $\gamma = 0.1$, and $\gamma = 0.01$ in their appropriate intervals of applicability, the actual time-dependent behavior of the function may be determined. Presumably, better approximations could be obtained by including more terms in the series expansions.

The magnitude of the coefficients of the Legendre polynomials limits the number of terms which can be considered in the series solution suggested by Papoulis. Since only the even Legendre polynomials are used, the order of the polynomials used is almost double the number of terms considered, e.g., a six-term expansion involves the use of a tenth-order polynomial. Since the magnitude of the coefficients of the Legendre polynomials increase very rapidly and small differences between very large numbers are significant as discussed earlier, there exists a practical limit to the number of terms that can be considered with the current capacities of high-speed computers. For the results presented in this paper, the Legendre polynomial expansions were obtained using double-precision computational schemes. Single precision routines were not of sufficient accuracy to give meaningful results.

In order to investigate the effect of slight error in the values of the transformed function which are used as data for the orthogonal expansion, an analysis was made for a series of exponentials of the form

$$f(t) = \sum_{i=0}^{5} e^{-2it} \tag{54}$$

Operating on this function with the Laplace transform with respect to time gives

$$\bar{f}(p) = \sum_{i=0}^{5} \left(\frac{1}{p + 2i}\right) \tag{55}$$

Values of this function were computed for p values of 1, 3, 5, 7, and 9 and the results used as input data for obtaining an orthogonal function expansion using Papoulis' Legendre polynomial inversion method. The results of this computation are shown in Fig. 9 as a solid line.

To examine the effect of error, the data point corresponding to $p = 7$ was changed from the correct value of 3.8310344

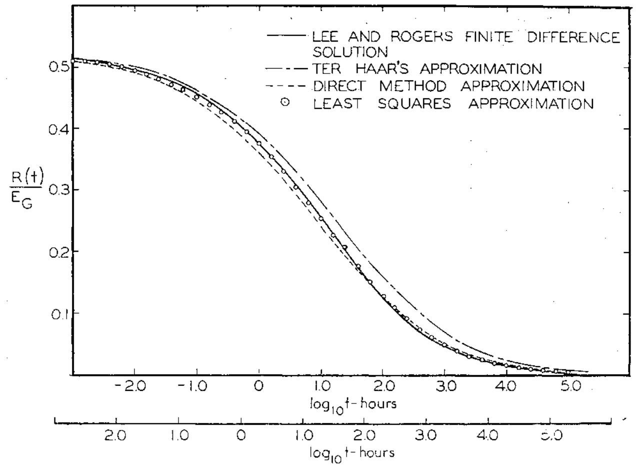

Fig. 8 Direct method, ter Haar's method, and eight term least-squares approximations for $R(t)/E_G$.

to 3.8348653, an error of 0.1%. The expansion was made in the same manner as before with the result shown by the dashed curve in Fig. 9. It can be seen that the method is extremely sensitive to errors in the original data, i.e., errors in values of the transformed function.

Conclusions

From the investigation conducted and results presented, several general conclusions can be drawn concerning the Laplace transform inversion methods discussed here.

In regard to Schapery's direct method and ter Haar's method, it appears that these methods, when applicable, will give a good description of the behavior of the time-dependent function. It is possible that the methods will give slight errors in magnitude at any instant of time. Also, the difference between the transform parameter p and time t is different for the two methods that result in the curves being slightly shifted from each other in the $\log t$ scale. Although Schapery's method for evaluating the relation between these parameters is more sophisticated than ter Haar's, ter Haar's method sometimes gives more accurate results.[12] In general, these methods provide a very quick and easy method for approximately inverting Laplace transforms, provided the functions are nearly linear in $\log t$. Large errors can occur if not.[12]

The methods of Widder which involve derivatives of successively higher orders appear to be of very limited use for approximately inverting functions known only numerically. The error involved with the use of numerical derivatives is sometimes great and hence makes the use of such methods highly questionable. If the method is applied to

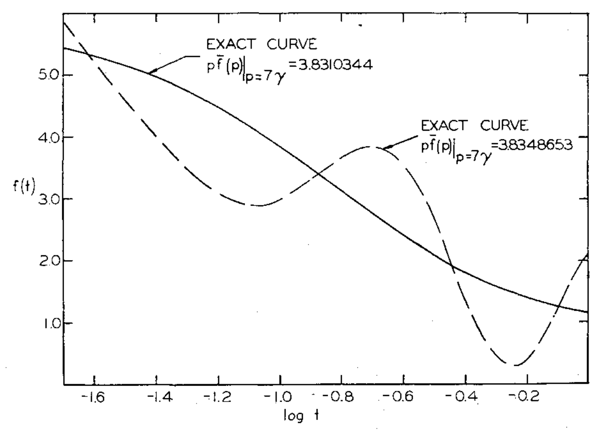

Fig. 9 Illustration of error sensitivity of Papoulis method.

Table 3 Coefficients of D_j in Eq. (A3) used in determining expression for Legendre polynomial series expansion

	D_0	D_1	D_2	D_3	D_4	D_5
B_0	1	0	0	0	0	0
B_1	−2.5	2.5	0	0	0	0
B_2	3.375	−11.25	7.875	0	0	0
B_3	−4.0625	28.4375	−51.1875	26.8125	0	0
B_4	4.6484375	−55.78125	184.078125	−227.90625	94.960903	0
B_5	−5.1679688	+99.746094	−164.22656	1055.7422	−997.08984	344.44922

problems with complicated algebraic analytical solutions where the derivatives of the transformed function can be determined exactly, then the methods give good approximations.

The collocation method or least-squares method of Schapery appears to offer a very good method for inverting functions of interest in viscoelastic stress analysis of solid propellant rocket grains. Of all the methods considered, the least-squares method seems to possess the greatest potential.

The orthogonal polynomial inversion methods are mathematically well founded and appear to possess many desirable features. The main disadvantage of the method is the extreme sensitivity to error in the original data. If this disadvantage could be overcome, the methods could be used to refine an answer to any degree of accuracy desired. In their present form, however, use of the methods appears limited.

By using a combination of the methods described in this paper, reliable results can be obtained to the problem of approximately inverting functions known only numerically for discrete values of the transform parameter. If accuracy of the magnitude being demanded in the representation of viscoelastic material property data is desired, only the orthogonal polynomial methods and Schapery's least-squares method possess the potential for accomplishing the inversion with this accuracy. The orthogonal polynomial methods appear limited by their sensitivity to error, but if this could be overcome, the results could be refined to any degree of accuracy desired.

Appendix: Simplified Form of Legendre Polynomial Inversion Method

The method of approximate Laplace transform inversion based upon properties of Legendre polynomials and described by Papoulis has been further simplified for the case of a six-term expansion. The simplification was made with the intention of programing the method for operation on a digital computer. Included in this Appendix is an expression for the time-dependent function in terms of values of the function $p\bar{f}(p)$, where $\bar{f}(p)$ is the Laplace transform of the time-dependent function of interest $f(t)$. The expression is presented in such a manner that errors due to rounding off are kept to a minimum. The expression for $f(t)$ is

$$f(t) = \left[B_0 - \frac{B_1}{2} + \frac{3}{8}B_2 - \frac{5}{16}B_3 + \frac{35}{128}B_4 - \frac{81}{256}B_i\right] + \frac{3}{2}\left[B_1 - \frac{5}{2}B_2 + \frac{35}{8}B_3 + \frac{85}{16}B_4 + \frac{935}{128}B_5\right]e^{-2\gamma t} + \frac{35}{8}\left[B_2 - \frac{9}{2}B_3 + \frac{99}{8}B_4 - \frac{429}{16}B_5\right]e^{-4\gamma t} + \frac{231}{16}\left[B_3 - \frac{13}{2}B_4 + \frac{195}{8}B_5\right]e^{-6\gamma t} + \frac{6435}{128}\left[B_4 - \frac{17}{2}B_5\right]e^{-8\gamma t} + \frac{46189}{256}[B_5]e^{-10\gamma t} \quad \text{(A1)}$$

where the B_i are given in terms of the values of $p\bar{f}(p)$ in Table 3. The D_i referred to in the table correspond to data points and are defined as

$$D_i = p\bar{f}(p)|_{p=[2i+1]\gamma} \qquad (i = 0, 1, \ldots, 5) \quad \text{(A2)}$$

The values given in the table are the coefficients α_{ij} of the D_i where the B_i are expressed as

$$B_i = \sum_{j=0}^{5} \alpha_{ij} D_j \quad \text{(A3)}$$

Equation (A1) was programed for the IBM 7090 digital computer and the program used to obtain the polynomial expansion results included in this paper. Equation (A1) may be used with less than six terms by simply neglecting the B_i, which correspond to the terms to be neglected.

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